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ON PAIRWISE ALMOST QUASI-CONTINUOUS MAPPINGS IN BITOPOLOGICAL SPACES

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ABSTRACT

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Almost quasi-continuity, introduced by Popa (1978) in the setting of classical topological spaces, has played an important role in the development of generalized continuity concepts. Motivated by the increasing interest in continuity notions within bitopological frameworks, this paper extends the concept of almost quasi-continuity to bitopological spaces through the introduction of pairwise almost quasi-continuous mappings. Several equivalent characterizations of this new class are established, and its fundamental properties are investigated. Furthermore, the relationships between pairwise almost quasi-continuity and other generalized pairwise continuity concepts are examined. As a main result, it is shown that pairwise almost quasi-continuity is equivalent to pairwise semi-weak continuity. These results enrich the theory of generalized continuity in bitopological spaces and provide a solid foundation for future studies of mappings between spaces equipped with two interacting topologies.

Keywords: (i, j)-regular open; (i, j)-semi open; pairwise almost quasi continuous.

1. INTRODUCTION

Topological approaches have become increasingly important in the study of uncertainty, approximation processes, and complex relational systems. Their flexibility and structural depth have led to wide-ranging applications in both pure and applied mathematics. In particular, topology-based techniques have played a substantial role in the development of rough set theory and several of its modern extensions (Abu-Gdairi *et al.*, 2023; El-Bably *et al.* 2024; El-Bably *et al.* 2025a; El-Bably *et al.* 2025b; R. A. Hosny *et al.* 2025).

Beyond their theoretical importance, topological models have also demonstrated considerable effectiveness in intelligent systems, data analysis, and decision-making frameworks (El-Gayar *et al.*, 2026; Nawar *et al.*, 2024; Nawar *et al.*, 2022). Recent studies further highlight their relevance in medical and healthcare applications, particularly in predictive diagnosis and feature selection problems related to diseases such as medical applications (Hosny, R. A. *et al.*, 2024; Abu-Gdairi & El-Bably, 2024; El-Bably *et al.*, 2021). Such developments reflect the growing reliance on topological frameworks for handling incomplete, vague, and uncertain information in complex environments. Consequently, increasing attention has been directed toward extending classical topological notions to settings capable of describing more flexible and multi-layered structures.

Bitopological spaces, introduced by J. C. Kelly (Kelly, 1963), constitute a natural extension of ordinary topological spaces by equipping a set with two interacting topologies. This framework has generated a broad and active research area due to its ability to capture structural phenomena that cannot be adequately represented within a single topology, for example see (Popa, V., & Noiri, T., 2004; Tahiliani, S., 2008; Kılıçman, A., & Salleh, Z., 2009; Ashaea, G.S., & Yaqoub, Y.,

2020) and Soft bitopological ordered spaces in (Shalil, S. H. *et al.*, 2024a; Shalil, S. H. *et al.*, 2024b; Shalil, S. H. *et al.*, 2024c). The interaction between the two topologies has led to numerous generalized forms of topological properties and mapping behaviors. Among the most extensively investigated topics in this context is generalized continuity (Nasef & Noiri, 1998, 2000). Several weakened and hybrid continuity notions have subsequently been studied, including pairwise continuity (Khedr & Al-Shibani, 1991), semi-continuity (Noiri & Ahmad, 1985), irresoluteness (Dube, Panwar & Tiwari, 1990), and almost continuity (Nasef & Noiri, 1997). These generalized forms provide useful analytical tools for examining mappings between spaces endowed with dual topological structures.

In a related direction, quasi-continuity and its various modifications have contributed significantly to the understanding of local mapping behavior beyond classical continuity. Almost quasi-continuity was originally introduced by Popa (1978) in topological spaces and was later investigated further by Noiri (Noiri & Ahmad, 1985; Noiri, 1990). This notion combines aspects of almost continuity and quasi-continuity, thereby offering a more flexible continuity-type condition. Although the concept has been studied extensively in classical topology, its counterpart in bitopological spaces has not yet received comparable attention. In particular, the interaction between almost quasi-continuity and pairwise continuity structures remains insufficiently explored.

Motivated by these observations, the present paper introduces and studies the notion of *pairwise almost quasi-continuous mappings* in bitopological spaces. The proposed concept is investigated through several equivalent formulations that clarify its essential structural characteristics. Moreover, comparisons with existing classes of pairwise continuity are established in order to determine its position

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within the hierarchy of generalized continuity notions.

The main contributions of this work may be summarized as follows. First, the concept of almost quasi-continuity is extended to the bitopological setting through the introduction of pairwise almost quasi-continuous mappings, thereby enriching the existing theory of generalized continuity in bitopological spaces. Second, multiple equivalent characterizations are obtained, providing alternative perspectives for understanding the behavior and structure of this class. Third, it is proved that pairwise almost quasi-continuity coincides with pairwise semi-weak continuity, leading to a unified interpretation of two continuity notions that initially appear to be different.

In addition to its theoretical significance, the proposed framework may also be useful in applications where bitopological structures arise naturally. Such situations frequently occur in rough set theory, particularly when lower and upper approximations are generated through distinct topological or relational mechanisms. Accordingly, the obtained results provide useful tools for studying mappings in environments characterized by uncertainty and incomplete information. Potential applications may also extend to decision support systems, granular computing, information processing, and data analysis.

The paper is organized as follows. Section 2 presents the basic definitions and preliminary results used throughout the paper. In Section 3, the concept of pairwise almost quasi-continuous mappings is introduced together with several equivalent characterizations. Section 4 investigates fundamental properties of this class, including behavior under composition, restriction, and product constructions. Finally, Section 5 concludes the paper and outlines several possible directions for future research.

2. Terminologies

Throughout this paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote arbitrary bitopological spaces.

For a subset $\mathfrak{A} \subseteq X$, the notations $\tau_i\text{-Cl}(\mathfrak{A})$ and $\tau_i\text{-Int}(\mathfrak{A})$ denote, respectively, the closure and interior of $\mathfrak{A}$ with respect to the topology τ_i , where $i \in \{1, 2\}$. Whenever no ambiguity arises, these notations will simply be written as $\text{Cl}_i(\mathfrak{A})$ and $\text{Int}_i(\mathfrak{A})$.

Throughout this paper, the indices i and j are assumed to satisfy $i, j \in \{1, 2\}$ and $i \neq j$, unless explicitly stated otherwise.

Definition 2.1. Let $\mathfrak{A}$ be a subset of a bitopological space (X, τ_1, τ_2) . Then $\mathfrak{A}$ is said to be:

- (a) (τ_i, τ_j) -regular open (Jelić, 1990a) (briefly, (i, j) -regular open) whenever

$$\mathfrak{A} = \text{Int}_i(\text{Cl}_j(\mathfrak{A})).$$

- (b) (τ_i, τ_j) -semi-open (Maheshwari & Prasad, 1977/1978) (briefly, (i, j) -semi-open) whenever

$$\mathfrak{A} \subseteq \text{Cl}_j(\text{Int}_i(\mathfrak{A})).$$

- (c) (τ_i, τ_j) -preopen (Jelić, 1990b) (briefly, (i, j) -preopen) whenever

$$\mathfrak{A} \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{A})).$$

- (d) (τ_i, τ_j) - α -open (Jelić, 1990b) (briefly, (i, j) - α -open) whenever

$$\mathfrak{A} \subseteq \text{Int}_i(\text{Cl}_j(\text{Int}_i(\mathfrak{A}))).$$

Definition 2.2. (Jelić, 1990a) A subset $\mathfrak{A}$ of a bitopological space (X, τ_1, τ_2) is called (τ_i, τ_j) - δ -open (or simply (i, j) - δ -open) if for every point $x \in \mathfrak{A}$ there exists an (i, j) -regular open set $\mathfrak{W}$ such that

$$x \in \mathfrak{W} \subseteq \mathfrak{A}.$$

The complement of an (i, j) -regular open set (respectively, an (i, j) -semi-open or (i, j) - δ -open set) is called an (i, j) -regular closed set (respectively, an (i, j) -semi-closed or (i, j) - δ -closed set).

For a bitopological space $X \equiv (X, \tau_1, \tau_2)$, the families of all (i, j) -regular open, (i, j) -regular closed, (i, j) -semi-open, (i, j) -semi-closed, and (i, j) -preopen subsets of X are denoted by

$$(i, j)\text{-RO}(X), \quad (i, j)\text{-RC}(X), \quad (i, j)\text{-SO}(X), \quad (i, j)\text{-SC}(X), \\ (i, j)\text{-PO}(X),$$

respectively.

A subset $\mathfrak{A}$ is termed *pairwise semi-open* if it is simultaneously $(1, 2)$ -semi-open and $(2, 1)$ -semi-open. Similarly, the notions of pairwise regular open, pairwise δ -open, pairwise regular closed, and pairwise δ -closed sets are defined in the obvious manner.

Definition 2.3. (Jelić, 1990a) Let $\mathfrak{A}$ be a subset of a bitopological space (X, τ_1, τ_2) . The intersection of all (i, j) -semi-closed sets containing $\mathfrak{A}$ is called the (τ_i, τ_j) -semi-closure of $\mathfrak{A}$. Likewise, the intersection of all (i, j) - δ -closed sets containing $\mathfrak{A}$ is called the (τ_i, τ_j) - δ -closure of $\mathfrak{A}$.

These are denoted, respectively, by

$$(\tau_i, \tau_j)\text{-sCl}(\mathfrak{A}) \quad \text{and} \quad (\tau_i, \tau_j)\text{-}\delta\text{Cl}(\mathfrak{A}).$$

Definition 2.4. (Pervin, 1967) A mapping

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$$

is said to be *pairwise continuous* (respectively, *pairwise open*) provided that both mappings

$$f : (X, \tau_1) \rightarrow (Y, \sigma_1) \quad \text{and} \quad f : (X, \tau_2) \rightarrow (Y, \sigma_2)$$

are continuous (respectively, open).

Definition 2.5. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called *pairwise semi-continuous* (Maheshwari & Prasad, (1977/1978)) (respectively, *pairwise irresolute* (Dube, Panwar, & Tiwari, 1990), or a *pairwise R-map* (Khedr & Al-Shibani, (1991))) if the inverse image of every σ_i -open set (respectively, every (i, j) -semi-open set or every (i, j) -regular open set) in (Y, σ_1, σ_2) is (i, j) -semi-open (respectively, (i, j) -semi-open or (i, j) -regular open) in (X, τ_1, τ_2) .

Definition 2.6. (Bose & Sinha, 1982) A mapping

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$$

is said to be *pairwise almost continuous* if for each point $x \in X$ and every σ_i -open neighborhood $\mathfrak{V}$ of $f(x)$, there exists a τ_i -open neighborhood $\mathfrak{U}$ of x satisfying

$$f(\mathfrak{U}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{V})).$$

Definition 2.7. (Bose & Sinha, 1982; Khedr, & Al-Shibani, 1991) A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be *pairwise quasi-continuous at a point* $x \in X$ provided that for every τ_i -open set $\mathfrak{U}$ containing x and every σ_i -open set $\mathfrak{V}$ containing $f(x)$, there exists a nonempty τ_i -open set $\mathfrak{G}$ in X such that

$$\mathfrak{G} \subseteq \mathfrak{U} \quad \text{and} \quad f(\mathfrak{G}) \subseteq \mathfrak{V}.$$

If the above condition holds for every point $x \in X$, then f is called *pairwise quasi-continuous*.

3. Characterizations

This section is devoted to establishing several equivalent descriptions of pairwise almost quasi-continuity. These formulations provide different viewpoints for studying this class of mappings and help reveal its underlying structural behavior.

Definition 3.1. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a mapping. Then f is called *pairwise almost quasi-continuous* (briefly, pairwise a.q.c.) at a point $x \in X$ if for every τ_i -open neighborhood $\mathfrak{U}$ of x and every σ_i -open neighborhood $\mathfrak{V}$ of $f(x)$, there exists a nonempty τ_i -open set $\mathfrak{G}$ such that

$$\mathfrak{G} \subseteq \mathfrak{U} \quad \text{and} \quad f(\mathfrak{G}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{V})).$$

A mapping f is called *pairwise almost quasi-continuous* if it is pairwise almost quasi-continuous at every point of X .

Theorem 3.2. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a mapping. Then the following conditions are equivalent:

(a) f is pairwise almost quasi-continuous;

(b) For every $x \in X$ and every σ_i -open set $\mathfrak{V}$ containing $f(x)$, there exists an (i, j) -semi-open set $\mathfrak{U}$ containing x such that

$$f(\mathfrak{U}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{V}));$$

(c) Whenever $x \in X$ and

$$\mathfrak{V} \in (i, j)\text{-RO}(Y)$$

satisfies $f(x) \in \mathfrak{V}$, there exists

$$\mathfrak{U} \in (i, j)\text{-SO}(X)$$

such that $x \in \mathfrak{U}$ and

$$f(\mathfrak{U}) \subseteq \mathfrak{V};$$

(d) For every subset $\mathfrak{B} \subseteq Y$,

$$(i, j)\text{-sCl}(f^{-1}(\text{Cl}_j(\text{Int}_i(\text{Cl}_j(\mathfrak{B})))) \subseteq f^{-1}(\text{Cl}_j(\mathfrak{B}));$$

(e) For every (i, j) -regular closed subset $\mathfrak{F}$ of Y , the inverse image

$$f^{-1}(\mathfrak{F})$$

is (i, j) -semi-closed in X ;

(f) For each (i, j) -semi-open subset $\mathfrak{V}$ of Y ,

$$(i, j)\text{-sCl}(f^{-1}(\mathfrak{V})) \subseteq f^{-1}(\text{Cl}_j(\mathfrak{V}));$$

(g) For every (i, j) -regular open set $\mathfrak{V}$ in Y , the inverse image

$$f^{-1}(\mathfrak{V})$$

is an (i, j) -semi-open subset of X .

Proof. (a) $\Rightarrow$ (b):

Let $x \in X$ and consider the family

$$\Sigma_x = \{\mathfrak{N} \subseteq X : x \in \mathfrak{N} \text{ and } \mathfrak{N} \text{ is } \tau_1\text{-open}\}.$$

Assume that $\mathfrak{V}$ is a σ_i -open subset of Y containing $f(x)$. Since f is pairwise a.q.c., for every $\mathfrak{N} \in \Sigma_x$ there exists a nonempty τ_1 -open set $\mathfrak{G}_{\mathfrak{N}}$ satisfying

$$\mathfrak{G}_{\mathfrak{N}} \subseteq \mathfrak{N} \text{ and } f(\mathfrak{G}_{\mathfrak{N}}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{V})).$$

Now define

$$\mathfrak{G} = \bigcup_{\mathfrak{N} \in \Sigma_x} \mathfrak{G}_{\mathfrak{N}}.$$

Clearly, $\mathfrak{G}$ is τ_1 -open and satisfies

$$x \in \text{Cl}_j(\mathfrak{G}).$$

Set

$$\mathfrak{U} = \mathfrak{G} \cup \{x\}.$$

Then

$$\mathfrak{G} \subseteq \mathfrak{U} \subseteq \text{Cl}_j(\mathfrak{G}),$$

and consequently $\mathfrak{U}$ is (i, j) -semi-open. Moreover, $x \in \mathfrak{U}$ and

$$f(\mathfrak{U}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{V})).$$

Hence condition (b) holds.

(b) $\Rightarrow$ (c):

The conclusion follows immediately from the relevant definitions.

(c) $\Rightarrow$ (d):

Let $\mathfrak{B} \subseteq Y$ and suppose that

$$x \notin f^{-1}(\text{Cl}_j(\mathfrak{B})).$$

Then

$$f(x) \notin \text{Cl}_j(\mathfrak{B}).$$

Accordingly, there exists a σ_i -open set $\mathfrak{V}$ containing $f(x)$ such that

$$\mathfrak{V} \cap \mathfrak{B} = \emptyset.$$

Therefore,

$$\text{Int}_i(\text{Cl}_j(\mathfrak{V})) \cap \text{Cl}_j(\text{Int}_i(\text{Cl}_j(\mathfrak{B}))) = \emptyset.$$

By condition (c), there exists a set

$$\mathfrak{U} \in (i, j)\text{-SO}(X)$$

such that $x \in \mathfrak{U}$ and

$$f(\mathfrak{U}) \cap \text{Cl}_j(\text{Int}_i(\text{Cl}_j(\mathfrak{B}))) = \emptyset.$$

Consequently,

$$\mathfrak{U} \cap f^{-1}(\text{Cl}_j(\text{Int}_i(\text{Cl}_j(\mathfrak{B})))) = \emptyset,$$

which implies that

$$x \notin (i, j)\text{-sCl}(f^{-1}(\text{Cl}_j(\text{Int}_i(\text{Cl}_j(\mathfrak{B}))))).$$

Hence,

$$(i, j)\text{-sCl}(f^{-1}(\text{Cl}_j(\text{Int}_i(\text{Cl}_j(\mathfrak{B})))) \subseteq f^{-1}(\text{Cl}_j(\mathfrak{B})).$$

(d) $\Rightarrow$ (e):

Let $\mathfrak{F}$ be an (i, j) -regular closed subset of Y . By condition (d),

$$(i, j)\text{-sCl}(f^{-1}(\mathfrak{F})) \subseteq f^{-1}(\mathfrak{F}).$$

Therefore,

$$f^{-1}(\mathfrak{F})$$

is (i, j) -semi-closed in X .

(e) $\Rightarrow$ (f):

Let

$$\mathfrak{V} \in (i, j)\text{-SO}(Y).$$

Since

$$\text{Cl}_j(\mathfrak{V})$$

is (i, j) -regular closed in Y , condition (e) implies that

$$f^{-1}(\text{Cl}_j(\mathfrak{V}))$$

is (i, j) -semi-closed in X . Hence,

$$(i, j)\text{-sCl}(f^{-1}(\mathfrak{V})) \subseteq f^{-1}(\text{Cl}_j(\mathfrak{V})).$$

(f) $\Rightarrow$ (g):

Let

$$\mathfrak{V} \in (i, j)\text{-RO}(Y).$$

Then

$$Y \setminus \mathfrak{V}$$

is (i, j) -regular closed and therefore (i, j) -semi-open. By applying condition (f), we obtain

$$(i, j)\text{-sCl}(f^{-1}(Y \setminus \mathfrak{V})) \subseteq f^{-1}(Y \setminus \mathfrak{V}).$$

Thus,

$$f^{-1}(Y \setminus \mathfrak{V})$$

is (i, j) -semi-closed, and consequently

$$f^{-1}(\mathfrak{V})$$

is (i, j) -semi-open.

(g) $\Rightarrow$ (a):

Let $x \in X$, let $\mathfrak{U} \in \tau_1$ with $x \in \mathfrak{U}$, and let $\mathfrak{V} \in \sigma_1$ such that

$$f(x) \in \mathfrak{V}.$$

Since

$$\text{Int}_i(\text{Cl}_j(\mathfrak{V})) \in (i, j)\text{-}RO(Y),$$

condition (g) yields

$$f^{-1}(\text{Int}_i(\text{Cl}_j(\mathfrak{V}))) \in (i, j)\text{-}SO(X).$$

Therefore,

$$x \in \mathfrak{U} \cap f^{-1}(\text{Int}_i(\text{Cl}_j(\mathfrak{V}))),$$

and this set is (i, j) -semi-open.

Define

$$\mathfrak{G} = \text{Int}_i(\mathfrak{U} \cap f^{-1}(\text{Int}_i(\text{Cl}_j(\mathfrak{V}))))).$$

Then

$$\mathfrak{G} \neq \emptyset, \quad \mathfrak{G} \in \tau_1, \quad \mathfrak{G} \subseteq \mathfrak{U},$$

and

$$f(\mathfrak{G}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{V})).$$

Hence, f is pairwise almost quasi-continuous. $\square$

Theorem 3.3. The following conditions are equivalent for a mapping

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) :$$

(a) f is pairwise almost quasi-continuous;

(b) For every subset $\mathfrak{A} \subseteq X$,

$$f((i, j)\text{-}sCl(\mathfrak{A})) \subseteq (i, j)\text{-}\delta Cl(f(\mathfrak{A}));$$

(c) For every subset $\mathfrak{B} \subseteq Y$,

$$(i, j)\text{-}sCl(f^{-1}(\mathfrak{B})) \subseteq f^{-1}((i, j)\text{-}\delta Cl(\mathfrak{B}));$$

(d) For every (i, j) - δ -closed subset $\mathfrak{F}$ of Y , the inverse image

$$f^{-1}(\mathfrak{F})$$

is (i, j) -semi-closed in X ;

(e) For every (i, j) - δ -open subset $\mathfrak{V}$ of Y , the inverse image

$$f^{-1}(\mathfrak{V})$$

is (i, j) -semi-open in X .

Proof. (a) $\Rightarrow$ (b):

Let $\mathfrak{A} \subseteq X$. Since

$$(i, j)\text{-}\delta Cl(f(\mathfrak{A}))$$

is (i, j) - δ -closed in Y , it may be represented as the intersection of a collection of (i, j) -regular closed sets. Thus,

$$(i, j)\text{-}\delta Cl(f(\mathfrak{A})) = \bigcap_{\alpha \in \Lambda} \mathfrak{F}_\alpha, \quad \mathfrak{F}_\alpha \in (i, j)\text{-}RC(Y).$$

Consequently,

$$\mathfrak{A} \subseteq f^{-1}((i, j)\text{-}\delta Cl(f(\mathfrak{A}))).$$

By Theorem 3.2, the set

$$f^{-1}((i, j)\text{-}\delta Cl(f(\mathfrak{A})))$$

is (i, j) -semi-closed in X . Therefore,

$$(i, j)\text{-}sCl(\mathfrak{A}) \subseteq f^{-1}((i, j)\text{-}\delta Cl(f(\mathfrak{A}))).$$

Applying f to both sides yields

$$f((i, j)\text{-}sCl(\mathfrak{A})) \subseteq (i, j)\text{-}\delta Cl(f(\mathfrak{A})).$$

(b) $\Rightarrow$ (c):

Let $\mathfrak{B} \subseteq Y$. Using condition (b) for the set

$$f^{-1}(\mathfrak{B}),$$

we obtain

$$f((i, j)\text{-}sCl(f^{-1}(\mathfrak{B}))) \subseteq (i, j)\text{-}\delta Cl(f(f^{-1}(\mathfrak{B}))).$$

Since

$$f(f^{-1}(\mathfrak{B})) \subseteq \mathfrak{B},$$

it follows that

$$(i, j)\text{-}\delta Cl(f(f^{-1}(\mathfrak{B}))) \subseteq (i, j)\text{-}\delta Cl(\mathfrak{B}).$$

Hence,

$$(i, j)\text{-}sCl(f^{-1}(\mathfrak{B})) \subseteq f^{-1}((i, j)\text{-}\delta Cl(\mathfrak{B})).$$

(c) $\Rightarrow$ (d):

Assume that $\mathfrak{F}$ is an (i, j) - δ -closed subset of Y . Then

$$(i, j)\text{-}\delta Cl(\mathfrak{F}) = \mathfrak{F}.$$

Using condition (c), we get

$$(i, j)\text{-}sCl(f^{-1}(\mathfrak{F})) \subseteq f^{-1}((i, j)\text{-}\delta Cl(\mathfrak{F})) = f^{-1}(\mathfrak{F}).$$

Therefore, $f^{-1}(\mathfrak{F})$ is (i, j) -semi-closed in X .

(d) $\Rightarrow$ (e):

Let $\mathfrak{V}$ be an (i, j) - δ -open subset of Y . Then

$$Y \setminus \mathfrak{V}$$

is (i, j) - δ -closed. By condition (d),

$$f^{-1}(Y \setminus \mathfrak{V}) = X \setminus f^{-1}(\mathfrak{V})$$

is (i, j) -semi-closed in X .

Consequently, $f^{-1}(\mathfrak{V})$ is (i, j) -semi-open.

(e) $\Rightarrow$ (a):

Let

$$\mathfrak{V} \in (i, j)\text{-}RO(Y).$$

Since every (i, j) -regular open set is (i, j) - δ -open, condition (e) implies that

$$f^{-1}(\mathfrak{V}) \in (i, j)\text{-}SO(X).$$

Therefore, Theorem 3.3 guarantees that f is pairwise almost quasi-continuous. $\square$

Theorem 3.4. The following conditions are equivalent for a mapping

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) :$$

(a) f is pairwise almost quasi-continuous;

(b) The mapping

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1^s, \sigma_2^s)$$

is pairwise quasi-continuous;

(c) The mapping

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1^s, \sigma_2^s)$$

is pairwise semi-continuous, where σ_i^s denotes the semi-regularization of σ_i for $i = 1, 2$.

Proof. The equivalence of (a) and (b) follows directly from the definition of pairwise almost quasi-continuity together with the characterization of almost open sets in terms of the semi-regularization topologies σ_1^s and σ_2^s .

To prove (b) $\Rightarrow$ (c), recall that every pairwise quasi-continuous mapping is pairwise semi-continuous. Since the codomain $(Y, \sigma_1^s, \sigma_2^s)$ is equipped with the semi-regularization topologies, the conclusion follows immediately from the standard inclusion between pairwise quasi-continuity and pairwise semi-continuity.

Finally, (c) $\Rightarrow$ (a) follows from the fact that every σ_i^s -semi-open set is represented by a regular open set of (Y, σ_i) . Consequently, the inverse image of each regular open set under f is pairwise semi-open in (X, τ_1, τ_2) , which is precisely the definition of pairwise almost quasi-continuity. $\square$

Lemma 3.5. Let $\mathfrak{A}$ be a subset of a bitopological space

$$(X, \tau_1, \tau_2).$$

Then

$$\text{Int}_i(\text{Cl}_i(\mathfrak{A})) \subseteq (i, j)\text{-sCl}(\mathfrak{A}).$$

Proof. The conclusion follows from the relation

$$\text{Cl}_j((i, j)\text{-sCl}(\mathfrak{A})) = \text{Cl}_j(\mathfrak{A}),$$

which is valid for every subset $\mathfrak{A} \subseteq X$, together with Remark 9 and Theorem 11 in (Maheshwari & Prasad, (1977/1978)). $\square$

Lemma 3.6. Let $\mathfrak{U}$ be a subset of a bitopological space (X, τ_1, τ_2) . Then

$$\mathfrak{U} \in (i, j)\text{-PO}(X) \text{ if and only if } \text{Int}_i(\text{Cl}_j(\mathfrak{U})) = (i, j)\text{-sCl}(\mathfrak{U}).$$

Proof. Assume that $\mathfrak{U} \in (i, j)\text{-PO}(X)$.

By Lemma 3.1, we already have

$$\text{Int}_i(\text{Cl}_j(\mathfrak{U})) \subseteq (i, j)\text{-sCl}(\mathfrak{U})$$

for every subset

$$\mathfrak{U} \subseteq X.$$

It remains to verify the opposite inclusion. Let

$$x \notin \text{Int}_i(\text{Cl}_j(\mathfrak{U})).$$

Then

$$x \in X \setminus \text{Int}_i(\text{Cl}_j(\mathfrak{U})).$$

Observe that

$$X \setminus \text{Int}_i(\text{Cl}_j(\mathfrak{U}))$$

is an (i, j) -semi-open subset of X . Since $\mathfrak{U}$ is (i, j) -preopen, we have

$$\mathfrak{U} \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{U})).$$

Hence,

$$\mathfrak{U} \cap (X \setminus \text{Int}_i(\text{Cl}_j(\mathfrak{U}))) = \emptyset.$$

Therefore,

$$x \notin (i, j)\text{-sCl}(\mathfrak{U}),$$

which implies

$$(i, j)\text{-sCl}(\mathfrak{U}) \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{U})).$$

Consequently,

$$\text{Int}_i(\text{Cl}_j(\mathfrak{U})) = (i, j)\text{-sCl}(\mathfrak{U}).$$

Conversely, suppose that

$$\text{Int}_i(\text{Cl}_j(\mathfrak{U})) = (i, j)\text{-sCl}(\mathfrak{U}).$$

Since

$$\mathfrak{U} \subseteq (i, j)\text{-sCl}(\mathfrak{U})$$

for every subset

$$\mathfrak{U} \subseteq X,$$

it follows that

$$\mathfrak{U} \subseteq \text{Int}_i(\text{Cl}_j(\mathfrak{U})).$$

Hence, $\mathfrak{U} \in (i, j)\text{-PO}(X)$. $\square$

Definition 3.7. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called *pairwise semi-weakly continuous* if, for every point $x \in X$ and every σ_1 -open set $\mathfrak{V} \subseteq Y$ containing $f(x)$, there exists an (i, j) -semi-open set $\mathfrak{U} \subseteq X$ such that $x \in \mathfrak{U}$ and $f(\mathfrak{U}) \subseteq (i, j)\text{-sCl}(\mathfrak{V})$.

Corollary 3.8. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise almost quasi-continuous if and only if it is pairwise semi-weakly continuous.

Proof. The conclusion is obtained immediately from Theorem 3.4 together with Lemma 3.6. $\square$

Remark 3.9. For a mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the implications illustrated in **Figure 1** are valid.

Nevertheless, these implications cannot, in general, be reversed. Counterexamples supporting this observation may be found in Example 2.1 of (Bose, 1981), Example 1 of (Bose & Sinha, 1982), as well as in Examples 3.10 and 3.11.



Figure 1: Logical implication relationships between pairwise continuity, pairwise semi-continuity, pairwise almost continuity, and pairwise almost quasi-continuity.

Example 3.10. Consider the sets

$$X = Y = \{a, b, c, d\},$$

equipped with the bitopologies

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\},$$

$$\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\},$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{a, c\}, Y\}, \quad \sigma_2 = \{\emptyset, \{b\}, \{b, d\}, Y\}.$$

Define

$$f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$$

to be the identity mapping.

One can verify directly that f is pairwise almost quasi-continuous. However, f is not pairwise almost continuous. Indeed, the set $\{a, c\}$ is $(1, 2)$ -regular open in Y , while its inverse image under f fails to be τ_1 -open in X . Consequently, pairwise almost quasi-continuity does not necessarily imply pairwise almost continuity.

Example 3.11. Let $X = \{a, b, c\}$ be equipped with the topologies

$$\tau_1 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}, \quad \tau_2 = \{\emptyset, \{b\}, \{b, c\}, X\},$$

and let

$$Y = \{0, 1, 2\}$$

carry the bitopologies

$$\sigma_1 = \{\emptyset, \{1\}, \{0, 1\}, Y\}, \quad \sigma_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, Y\}.$$

Consider the mapping

$$f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$$

defined by

$$f(a) = f(c) = 1, \quad f(b) = 0.$$

It is straightforward to check that f satisfies the condition of pairwise almost quasi-continuity. Nevertheless, f is not pairwise semi-continuous. In fact, the singleton $\{1\}$ is σ_2 -open, whereas $f^{-1}(\{1\}) = \{a, c\}$, which is not $(2, 1)$ -semi-open in X . Therefore, pairwise almost quasi-continuity is strictly weaker than pairwise semi-continuity.

4. Basic Properties

Theorem 4.1. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a mapping, and consider the graph map

$$g : (X, \tau_1, \tau_2) \rightarrow (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$$

defined by

$$g(x) = (x, f(x)) \quad \text{for all } x \in X.$$

Then g is pairwise almost quasi-continuous if and only if f is pairwise almost quasi-continuous.

Proof. Let $x \in X$, and let $\mathfrak{V}$ be a σ_1 -open subset of Y containing $f(x)$. Then $X \times \mathfrak{V}$ is $(\tau_1 \times \sigma_1)$ -open in $X \times Y$, and clearly contains $g(x)$.

Assume first that g is pairwise almost quasi-continuous. By Corollary 3.8, there exists an (i, j) -semi-open set

$$\mathfrak{U} \subseteq X$$

such that

$$x \in \mathfrak{U}$$

and

$$g(\mathfrak{U}) \subseteq (i, j)\text{-sCl}(X \times \mathfrak{V}).$$

Since

$$(i, j)\text{-sCl}(X \times \mathfrak{V}) \subseteq X \times (i, j)\text{-sCl}(\mathfrak{V}),$$

it follows immediately that

$$f(\mathfrak{U}) \subseteq (i, j)\text{-sCl}(\mathfrak{V}).$$

Therefore, f is pairwise almost quasi-continuous.

Conversely, suppose that f is pairwise almost quasi-continuous. Let $\mathfrak{W}$ be an (i, j) -regular open subset of $X \times Y$ containing $g(x)$.

Then there exist sets

$$\mathfrak{U}_1 \in (i, j)\text{-RO}(X)$$

and

$$\mathfrak{V} \in (i, j)\text{-RO}(Y)$$

such that

$$(x, f(x)) \in \mathfrak{U}_1 \times \mathfrak{V} \subseteq \mathfrak{W}.$$

Using Theorem 3.2, there exists an (i, j) -semi-open subset

$$\mathfrak{U}_2 \subseteq X$$

satisfying

$$x \in \mathfrak{U}_2$$

and

$$f(\mathfrak{U}_2) \subseteq \mathfrak{V}.$$

Define

$$\mathfrak{U} = \mathfrak{U}_1 \cap \mathfrak{U}_2.$$

Then $\mathfrak{U}$ is an (i, j) -semi-open subset of X containing x . Furthermore,

$$g(\mathfrak{U}) \subseteq \mathfrak{U}_1 \times \mathfrak{V} \subseteq \mathfrak{W}.$$

Hence, g is pairwise almost quasi-continuous. $\square$

Theorem 4.2. Let

$$f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \quad \text{and} \quad g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \theta_1, \theta_2)$$

be two mappings. Then the composite mapping

$$g \circ f : (X, \tau_1, \tau_2) \longrightarrow (Z, \theta_1, \theta_2)$$

is pairwise almost quasi-continuous provided that one of the following conditions holds:

- f is pairwise almost quasi-continuous and g is a pairwise R -map;
- f is pairwise almost quasi-continuous and g is pairwise almost continuous;
- f is pairwise irresolute and g is pairwise almost quasi-continuous.

Proof. We prove each part separately.

- Assume that f is pairwise almost quasi-continuous and g is a pairwise R -map. Let W be a regular open set in (Z, θ_i) . Since g is a pairwise R -map, $g^{-1}(W)$ is regular open in (Y, σ_i) . Because f is pairwise almost quasi-continuous,

$$f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$$

is τ_i -semi-open. Hence, $g \circ f$ is pairwise almost quasi-continuous.

- Suppose that f is pairwise almost quasi-continuous and g is pairwise almost continuous. Since every pairwise almost continuous mapping is a pairwise R -map, the desired conclusion follows immediately from part (a).

- Assume that f is pairwise irresolute and g is pairwise almost quasi-continuous. Let W be a regular open set in (Z, θ_i) . Then $g^{-1}(W)$ is pairwise semi-open in (Y, σ_1, σ_2) . Since f is pairwise irresolute,

$$f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$$

is pairwise semi-open in (X, τ_1, τ_2) . Therefore, $g \circ f$ is pairwise almost quasi-continuous.

This completes the proof. $\square$

Lemma 4.3. Let (X, τ_1, τ_2) be a bitopological space. Suppose that

$$\mathfrak{A} \in (i, j)\text{-}PO(X) \quad \text{and} \quad \mathfrak{B} \in (i, j)\text{-}SO(X).$$

Then

$$\mathfrak{A} \cap \mathfrak{B} \in (i, j)\text{-}SO(\mathfrak{A}).$$

Proof. Since $\mathfrak{B}$ is (i, j) -semi-open in X , we have

$$\mathfrak{B} \subseteq \text{Cl}_j(\text{Int}_i(\mathfrak{B})).$$

Hence,

$$\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{A} \cap \text{Cl}_j(\text{Int}_i(\mathfrak{B})).$$

Since $\mathfrak{A}$ is (i, j) -preopen, the subspace topology on $\mathfrak{A}$ preserves (i, j) -semi-openness of intersections with (i, j) -semi-open sets. Therefore,

$$\mathfrak{A} \cap \text{Int}_i(\mathfrak{B})$$

is τ_i -open in the subspace $\mathfrak{A}$, and consequently,

$$\mathfrak{A} \cap \mathfrak{B} \subseteq \text{Cl}_{\mathfrak{A}}(\text{Int}_{\mathfrak{A}}(\mathfrak{A} \cap \mathfrak{B})).$$

Thus,

$$\mathfrak{A} \cap \mathfrak{B} \in (i, j)\text{-}SO(\mathfrak{A}).$$

□

Theorem 4.4. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise almost quasi-continuous mapping, and let X_0 be an (i, j) -preopen subset of (X, τ_1, τ_2) , where $i \neq j$ and $i, j \in \{1, 2\}$. Then the restriction

$$f|_{X_0} : (X_0, \tau_1|_{X_0}, \tau_2|_{X_0}) \rightarrow (Y, \sigma_1, \sigma_2)$$

is pairwise almost quasi-continuous.

Proof. Let

$$\mathfrak{V} \in (i, j)\text{-}RO(Y).$$

Since f is pairwise almost quasi-continuous, it follows from Theorem 3.2 that

$$f^{-1}(\mathfrak{V}) \in (i, j)\text{-}SO(X).$$

Consider now the inverse image under the restricted mapping. Then

$$(f|_{X_0})^{-1}(\mathfrak{V}) = f^{-1}(\mathfrak{V}) \cap X_0.$$

Since X_0 is (i, j) -preopen in X and $f^{-1}(\mathfrak{V})$ is (i, j) -semi-open in X , Lemma 4.3 implies that

$$f^{-1}(\mathfrak{V}) \cap X_0$$

is (i, j) -semi-open in the subspace X_0 . Hence,

$$(f|_{X_0})^{-1}(\mathfrak{V}) \in (i, j)\text{-}SO(X_0).$$

Therefore, $f|_{X_0}$ is pairwise almost quasi-continuous. □

Lemma 4.5. (Maheshwari & Prasad, (1977/1978)) Let (X, τ_1, τ_2) be a bitopological space. If $\mathfrak{A} \in (i, j)\text{-}SO(\mathfrak{B})$ and $\mathfrak{B}$ is τ_i -open in X , then $\mathfrak{A} \in (i, j)\text{-}SO(X)$.

Theorem 4.6. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function, and let $(\mathfrak{U}_\alpha)_{\alpha \in \nabla}$ be a τ_i -open cover of X . Suppose that for each $\alpha \in \nabla$, the restriction

$$f|_{\mathfrak{U}_\alpha} : (\mathfrak{U}_\alpha, \tau_1|_{\mathfrak{U}_\alpha}, \tau_2|_{\mathfrak{U}_\alpha}) \rightarrow (Y, \sigma_1, \sigma_2)$$

is pairwise almost quasi-continuous. Then f is pairwise almost quasi-continuous.

Proof. Let

$$\mathfrak{V} \in (i, j)\text{-}RO(Y).$$

Since $(\mathfrak{U}_\alpha)_{\alpha \in \nabla}$ is a cover of X , we obtain

$$f^{-1}(\mathfrak{V}) = X \cap f^{-1}(\mathfrak{V}) = \bigcup_{\alpha \in \nabla} (\mathfrak{U}_\alpha \cap f^{-1}(\mathfrak{V})).$$

Equivalently,

$$f^{-1}(\mathfrak{V}) = \bigcup_{\alpha \in \nabla} (f|_{\mathfrak{U}_\alpha})^{-1}(\mathfrak{V}).$$

Since each restriction $f|_{\mathfrak{U}_\alpha}$ is pairwise almost quasi-continuous, Theorem 3.2 implies that

$$(f|_{\mathfrak{U}_\alpha})^{-1}(\mathfrak{V}) \in (i, j)\text{-}SO(\mathfrak{U}_\alpha), \quad \forall \alpha \in \nabla.$$

Moreover, each $\mathfrak{U}_\alpha$ is τ_i -open in X . Hence, by Lemma 4.3,

$$(f|_{\mathfrak{U}_\alpha})^{-1}(\mathfrak{V}) \in (i, j)\text{-}SO(X), \quad \forall \alpha \in \nabla.$$

Since arbitrary unions of (i, j) -semi-open sets are (i, j) -semi-open, it follows that

$$f^{-1}(\mathfrak{V}) \in (i, j)\text{-}SO(X).$$

Therefore, by Theorem 3.2, f is pairwise almost quasi-continuous. □

Theorem 4.7. Let

$$f : \left(\prod_{\alpha \in \nabla} X_\alpha, \prod \tau_1(\alpha), \prod \tau_2(\alpha) \right) \rightarrow \left(\prod_{\alpha \in \nabla} Y_\alpha, \prod \sigma_1(\alpha), \prod \sigma_2(\alpha) \right)$$

be the product mapping defined by

$$f((x_\alpha)_{\alpha \in \nabla}) = (f_\alpha(x_\alpha))_{\alpha \in \nabla}.$$

If each mapping

$$f_\alpha : (X_\alpha, \tau_1(\alpha), \tau_2(\alpha)) \rightarrow (Y_\alpha, \sigma_1(\alpha), \sigma_2(\alpha))$$

is pairwise almost quasi-continuous, then the product mapping f is pairwise almost quasi-continuous.

Proof. Let

$$x = (x_\alpha)_{\alpha \in \nabla} \in \prod_{\alpha \in \nabla} X_\alpha,$$

and let

$$\mathfrak{V} \in (i, j)\text{-}RO\left(\prod_{\alpha \in \nabla} Y_\alpha\right)$$

such that $f(x) \in \mathfrak{V}$. Since the product topology admits a base consisting of finite products, there exist indices

$$\alpha(1), \alpha(2), \dots, \alpha(n) \in \nabla$$

and sets

$$\mathfrak{V}_{\alpha(K)} \in (\sigma_i(\alpha(K)), \sigma_j(\alpha(K)))\text{-}RO(Y_{\alpha(K)})$$

for

$$K = 1, 2, \dots, n,$$

such that

$$f(x) \in \left(\prod_{K=1}^n \mathfrak{V}_{\alpha(K)} \right) \times \left(\prod_{\alpha \notin \{\alpha(1), \dots, \alpha(n)\}} Y_\alpha \right) \subseteq \mathfrak{V}.$$

For each $K = 1, 2, \dots, n$, the mapping $f_{\alpha(K)}$ is pairwise almost quasi-continuous. Hence, by Theorem 3.2, $f_{\alpha(K)}^{-1}(\mathfrak{V}_{\alpha(K)})$ is $(\tau_i(\alpha(K)), \tau_j(\alpha(K)))$ -semiopen in $X_{\alpha(K)}$.

Define

$$\mathfrak{U} = \left(\prod_{K=1}^n f_{\alpha(K)}^{-1}(\mathfrak{V}_{\alpha(K)}) \right) \times \left(\prod_{\alpha \notin \{\alpha(1), \dots, \alpha(n)\}} X_\alpha \right).$$

By Lemma 4.3, the set $\mathfrak{U}$ is (i, j) -semiopen in

$$\prod_{\alpha \in \nabla} X_{\alpha}.$$

Moreover, $x \in \mathfrak{U}$, and for every $u = (u_{\alpha})_{\alpha \in \nabla} \in \mathfrak{U}$, we have

$$f_{\alpha(K)}(u_{\alpha(K)}) \in \mathfrak{V}_{\alpha(K)}, \quad K = 1, 2, \dots, n.$$

Therefore,

$$f(u) \in \left(\prod_{K=1}^n \mathfrak{V}_{\alpha(K)} \right) \times \left(\prod_{\alpha \notin \{\alpha(1), \dots, \alpha(n)\}} Y_{\alpha} \right) \subseteq \mathfrak{V}.$$

Consequently,

$$f(\mathfrak{U}) \subseteq \mathfrak{V}.$$

Thus, for every

$$x \in \prod_{\alpha \in \nabla} X_{\alpha}$$

and every

(i, j) -regular open set $\mathfrak{V}$

containing

$$f(x),$$

there exists an

(i, j) -semiopen set $\mathfrak{U}$

containing

$$x$$

such that

$$f(\mathfrak{U}) \subseteq \mathfrak{V}.$$

Hence, by Theorem 3.2, the product mapping f is pairwise almost quasi-continuous. $\square$

Theorem 4.8. *The product mapping f is pairwise almost quasi-continuous if and only if every coordinate mapping f_{α} is pairwise almost quasi-continuous for each $\alpha \in \nabla$.*

Proof. We first establish the necessity.

Suppose that f is pairwise almost quasi-continuous and choose an arbitrary index $\alpha \in \nabla$. Let $\mathfrak{V}_{\alpha}$ be a $(\sigma_1(\alpha), \sigma_2(\alpha))$ -regular open subset of Y_{α} . Consider the cylinder set

$$\mathfrak{V} = \mathfrak{V}_{\alpha} \times \prod_{\beta \neq \alpha} Y_{\beta}.$$

Since regular openness is preserved by this construction, $\mathfrak{V}$ is (σ_1, σ_2) -regular open in $\prod_{\gamma \in \nabla} Y_{\gamma}$. The pairwise almost quasi-continuity of f yields that

$$f^{-1}(\mathfrak{V})$$

is (τ_1, τ_2) -semiopen in $\prod_{\gamma \in \nabla} X_{\gamma}$.

On the other hand,

$$f^{-1}(\mathfrak{V}) = f_{\alpha}^{-1}(\mathfrak{V}_{\alpha}) \times \prod_{\beta \neq \alpha} X_{\beta}.$$

Applying Lemma 4.3, we conclude that

$$f_{\alpha}^{-1}(\mathfrak{V}_{\alpha})$$

is $(\tau_1(\alpha), \tau_2(\alpha))$ -semiopen in X_{α} . Therefore, f_{α} is pairwise almost quasi-continuous.

For the converse, assume that every coordinate function f_{α} is pairwise almost quasi-continuous. Let $(x_{\alpha}) \in \prod_{\alpha \in \nabla} X_{\alpha}$ and let W be a (σ_1, σ_2) -regular open neighborhood of $f((x_{\alpha}))$.

By the description of basic neighborhoods in the product topology, one can find a finite set $\nabla_0 \subseteq \nabla$ and $(\sigma_1(\gamma), \sigma_2(\gamma))$ -regular open sets $\mathfrak{V}_{\gamma} \subseteq Y_{\gamma}$, $\gamma \in \nabla_0$, satisfying

$$f((x_{\alpha})) \in \prod_{\gamma \in \nabla_0} \mathfrak{V}_{\gamma} \times \prod_{\beta \notin \nabla_0} Y_{\beta} \subseteq W.$$

For each $\gamma \in \nabla_0$, the pairwise almost quasi-continuity of f_{γ} provides a $(\tau_1(\gamma), \tau_2(\gamma))$ -semiopen neighborhood $\mathfrak{U}_{\gamma}$ of x_{γ} such that

$$f_{\gamma}(\mathfrak{U}_{\gamma}) \subseteq \mathfrak{V}_{\gamma}.$$

Now define

$$\mathfrak{U} = \prod_{\gamma \in \nabla_0} \mathfrak{U}_{\gamma} \times \prod_{\beta \notin \nabla_0} X_{\beta}.$$

Lemma 4.3 ensures that $\mathfrak{U}$ is (τ_1, τ_2) -semiopen in the product space, and clearly $(x_{\alpha}) \in \mathfrak{U}$.

Moreover,

$$f(\mathfrak{U}) \subseteq \prod_{\gamma \in \nabla_0} f_{\gamma}(\mathfrak{U}_{\gamma}) \times \prod_{\beta \notin \nabla_0} Y_{\beta} \subseteq W.$$

Hence every (σ_1, σ_2) -regular open neighborhood of $f((x_{\alpha}))$ admits a (τ_1, τ_2) -semiopen neighborhood of (x_{α}) mapped into it. Consequently, f is pairwise almost quasi-continuous. $\square$

Theorem 4.9. *Let $\{(Y_{\alpha}, \sigma_1(\alpha), \sigma_2(\alpha)) : \alpha \in \nabla\}$ be a family of bitopological spaces.*

If

$$f : (X, \tau_1, \tau_2) \rightarrow \left(\prod_{\alpha \in \nabla} Y_{\alpha}, \prod \sigma_1(\alpha), \prod \sigma_2(\alpha) \right)$$

is pairwise almost quasi-continuous, then for every $\alpha \in \nabla$, the mapping

$$\mathbb{P}_{\alpha} \circ f : (X, \tau_1, \tau_2) \rightarrow (Y_{\alpha}, \sigma_1(\alpha), \sigma_2(\alpha))$$

is pairwise almost quasi-continuous, where $\mathbb{P}_{\alpha}$ denotes the canonical projection onto Y_{α} .

Proof. Fix an arbitrary index $\alpha \in \nabla$, and let $\mathfrak{V}_{\alpha}$ be a $(\sigma_1(\alpha), \sigma_2(\alpha))$ -regular open subset of Y_{α} .

Since the projection $\mathbb{P}_{\alpha}$ is pairwise continuous and pairwise open, the inverse image

$$\mathbb{P}_{\alpha}^{-1}(\mathfrak{V}_{\alpha})$$

forms a $(\prod \sigma_1(\alpha), \prod \sigma_2(\alpha))$ -regular open subset of the product space $\prod_{\beta \in \nabla} Y_{\beta}$.

By the pairwise almost quasi-continuity of f , the set

$$f^{-1}(\mathbb{P}_{\alpha}^{-1}(\mathfrak{V}_{\alpha}))$$

is (τ_1, τ_2) -semiopen in X .

Using the identity

$$f^{-1}(\mathbb{P}_{\alpha}^{-1}(\mathfrak{V}_{\alpha})) = (\mathbb{P}_{\alpha} \circ f)^{-1}(\mathfrak{V}_{\alpha}),$$

we obtain that

$$(\mathbb{P}_{\alpha} \circ f)^{-1}(\mathfrak{V}_{\alpha})$$

is (τ_1, τ_2) -semiopen in X .

As $\mathfrak{V}_{\alpha}$ was chosen arbitrarily among the $(\sigma_1(\alpha), \sigma_2(\alpha))$ -regular open subsets of Y_{α} , the defining condition of pairwise almost quasi-continuity is satisfied. Hence, $\mathbb{P}_{\alpha} \circ f$ is pairwise almost quasi-continuous. $\square$

5. Conclusion

This paper introduced and systematically investigated the class of pairwise almost quasi-continuous mappings in bitopological spaces, thereby extending the classical notion of almost quasi-continuity to settings equipped with two interacting topologies. This extension provides a broader framework for studying generalized continuity and offers new insights into the interplay between bitopological structures.

Several equivalent characterizations of pairwise almost quasi-continuous mappings were established, leading to a deeper understanding of their structural properties. In addition, the relationships between this class and other generalized forms of pairwise continuity were thoroughly investigated. One of the main contributions of this work is the proof that pairwise almost quasi-continuity is equivalent to pairwise semi-weak continuity, thereby providing a unified perspective on these two notions.

Beyond its theoretical significance, the proposed framework may also contribute to the study of mathematical models involving dual topological structures. In particular, it has potential applications in rough set theory, where approximation operators frequently interact with topological or relational structures, as well as in information systems, granular computing, and data analysis, where multiple notions of continuity naturally arise.

Future research may focus on extending pairwise almost quasi-continuous mappings to more general mathematical environments, including fuzzy, nano, and other generalized topological spaces. Another promising direction is to investigate their role in advanced rough set models, decision-support systems, knowledge discovery, and information processing. Such extensions are expected to deepen the theoretical foundations of generalized continuity while expanding its applicability to computational and data-oriented disciplines.

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Conflicts of interest

Ethical approval was not required for this manuscript, as the study did not involve animals or human participants.

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